ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 281–287

A GENERALIZATION OF A RESULT OF R. LYONS ABOUT MEASURES ON [0,1)

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ABSTRACT

Let μ be a probability measure on $[0,1)$, invariant under $S\colon x \mapsto px \mod 1$, and for which almost every ergodic component has positive entropy. If q is a real number greater than 1 for which $\log q/\log p$ is irrational, and T_n sends x to $q^n x \mod 1$, then for any $\epsilon > 0$ the measure μT_n^{-1} will — for a set of n of positive lower density — be within ϵ of Lebesgue measure.

1. Introduction

The following fact is proved:

1.1. PROPOSITION: *Suppose p is an integer greater than one, and μ a probability measure on $[0,1)$ which is invariant under $S : x \mapsto px \bmod 1$ and has no S-invariant summand of zero entropy. Then for any $\epsilon > 0$ there is a positive integer b so that if v is a real number greater than one, and for some positive integer m we have $p^m b \le v < p^m(b+1)$, then setting $Vx = vx \bmod 1$, the measure μV^{-1} is within ϵ of Lebesgue measure (with respect to a preassigned metric for the weak $*$ topology).*

1.2. COROLLARY: *If p and μ are as in Theorem 1.1, and q is a real number greater than one for which $\log q/\log p$ is irrational, and $T_n x = q^n x \bmod 1$, then μT_n^{-1} is weak $*$ within ϵ of Lebesgue measure for a set of n of positive lower density. Consequently there is a subsequence of the sequence μT_n^{-1} which converges weak $*$ to Lebesgue measure.*

Received November 3, 1991 and in revised form July 28, 1992

When q is also an integer we denote by T the map $x \mapsto qx \bmod 1$; then what was called T_n is just T^n, and we are in the context of the paper [F] of H. Furstenburg. The result of Lyons alluded to in the title, Theorem 4 of [L], is the special case of Corollary 1.2 where q is an integer relatively prime to p and μ is *exact*. Indeed, the argument of Theorem 1.1 was motivated by Lyons' argument.

Corollary 1.2 immediately yields the following result of D. Rudolph [R] and A. Johnson [J]:

1.3. COROLLARY: *If p and q are integers greater than one and having no common power, and μ is a probability measure on [0,1) invariant under the corresponding transformations S and T, and μ has no S and T invariant summand of zero entropy under S, then μ is Lebesgue measure.*

This is clear because S and T commute, so that the zero-entropy component of μ under S is also invariant under T, and therefore has measure zero. We believe that others, among them J-P. Thouvenot, are aware of a proof of the Rudolph-Johnson result along similar lines, at least for the case of relatively prime p and q. Also, Rudolph has told us that he and Johnson, using their aforementioned theorem, can show that (for the case of integer q) the set of n described in Theorem 1.1 actually has lower density one.

Again for integer q, in the special case when the p-digit process is a nondegenerate i.i.d process, or more generally *weak Bernoulli* (see [F-O]), a stronger sort of convergence holds; in fact, μ *-a.e.x* is normal to the base q, i.e the sequence $q^n x \bmod 1$ is equidistributed on the interval (see [S], [K], [F-S]). It is conceivable that this remains true even when q is not assumed to be an integer. Indeed, if μ is Lebesgue measure this is well-known (see [W]). However, we will not pursue the question here.

The author is grateful to M. Smorodinsky for valuable conversations; to the participants in his Fall 1990 seminar at Berkeley for their suggestions and patience; and to the National Science Foundation for partial support (grant DMS 90-08102).

2. Preliminaries

The object of the first lemma is to show how Corollary 1.2 follows from Proposition 1.1.

2.1. LEMMA: *If q is a real number and p an integer, both greater than one, and $\log q/\log p$ is irrational, then for any positive integer b the set A of positive integers n for which $q^n = p^m b + c$ with m a positive integer and $0 \le c < p^m$ is a set of positive lower density.*

Proof: The interval $[\log b/\log p, \log(b+1)/\log p)$ may be shifted by an integer k to contain a nondegenerate subinterval I of $[0,1)$. By [W], the sequence $(n \log q/\log p) \bmod 1$ is equidistributed, so the set of positive n for which $n\log q/\log p$ lies in $j + I$ for some integer j is a set of positive lower density, and consequently likewise the set of positive n for which $n \log q$ lies in $m \log p + [\log b, \log(b+1))$ for some positive integer m. But this is precisely the set A. ∎

2.2. LEMMA: *Given $\epsilon > 0$ and a natural number K, and sets $A_1, ..., A_K$ of measure $> 1-\epsilon$ in a probability space, then the set of points which lie in no more than $K/2$ of the sets has measure less than 2ϵ.*

Proof: This is an easy Chebyshev inequality sort of argument, which is left to the reader. ∎

In the context of Proposition 1.1: denote the partition $[0,1/p), ..., [(p-1)/p, 1)$ by $\mathcal{P}$, and let $\mathcal{B}$ be the Borel subsets of $[0,1)$ completed by μ. Let $\mathcal{B}_n = S^{-n}\mathcal{B}$; then $\bigcap_n \mathcal{B}_n = \mathcal{B}_\infty$ is the so-called Pinsker algebra of S. Note that $\mathcal{B} = \bigvee_{n=0}^\infty S^{-n}\mathcal{P}$, so that $\mathcal{B}_\infty = \bigcap_{k=1}^\infty \bigvee_{n=k}^\infty S^{-n}\mathcal{P}$. Let P_n and P_∞ be the conditional expectations given $\mathcal{B}_n$ and $\mathcal{B}_\infty$ respectively; these may also be viewed as projection operators on $\mathcal{L}^2(\mu)$.

2.3. LEMMA: *Let $\phi_r(x) = e^{2i\pi r x}$ on $[0,1)$. If $r \ne 0$ then $|P_\infty \phi_r|$ cannot equal one on a set of positive measure.*

Proof: By using complex conjugation, we see that it suffices to show this for *positive* r. The Pinsker algebra, being invariant, gives rise to a factor map $\theta : [0,1) \mapsto Y$. The measure μ decomposes: $\mu(A) = \int_Y \mu_y(A)\, d\mu \circ \theta^{-1}(y)$, S pushes down to a transformation S_0 on Y, and uniqueness of the decomposition gives $\mu_{S_0 y} = \mu_y S^{-1}$ for $\mu \circ \theta^{-1}$ almost every y. The projection P_∞ is obtained simply by averaging with respect to the fibre measures: $P_\infty f(x) = \int_X f d\mu_{\theta x}$. For any x such that this has absolute value 1, $\mu_{\theta x}$ must be supported on a level set of ϕ_r, so $\mu_{\theta x}$ must consist of $\le r$ atoms. Let E be the set of all y in Y for which μ_y consists of $\le r$ atoms. Then E is invariant under S_0. But S_0, the restriction of S to its Pinsker algebra, has no invariant summand of positive entropy; thus the restriction of S_0 to E is of entropy zero, and since the restriction of S to E is a finite extension of this, it is likewise of entropy zero. Then the entropy assumption tells us that E must have measure zero. ∎

Let $\tilde{S}$ on $(X, \tilde{\mathcal{A}}, \tilde{\mu})$ be the canonical one -to-one extension of S. That is, $\tilde{S}$ is $\tilde{\mu}$-preserving and one-one on a set of full measure, and we have a map $\xi : X \mapsto [0,1)$ so that $\xi^{-1}\mathcal{B} \subset \tilde{\mathcal{A}}$, $\tilde{\mu} \circ \xi^{-1} = \mu$, $\xi\tilde{S} = S\xi$, and $\tilde{\mathcal{A}}$ is generated by $\xi^{-1}\mathcal{B}$ under $\tilde{S}$. Notice that

$$\xi^{-1}\mathcal{B}_\infty = \bigcap_{n=0}^{\infty} \tilde{S}^{-n}(\xi^{-1}\mathcal{B}) = \bigcap_{n=0}^{\infty} \bigvee_{k=n}^{\infty} \tilde{S}^{-k}(\xi^{-1}\mathcal{P}),$$

which is precisely the Pinsker algebra of $\tilde{S}$. Define $\tilde{\mathcal{C}}$ as $\bigvee_{n=-\infty}^{-1} \tilde{S}^{-n}(\xi^{-1}\mathcal{P})$, $\tilde{\mathcal{C}}_n$ as $\tilde{S}^{-n}\tilde{\mathcal{C}}$, and $\tilde{\mathcal{C}}_{-\infty}$ as $\bigcap_{n=-\infty}^{-1} \tilde{\mathcal{C}}_n$. Then $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}_n$ are not pullbacks via ξ^{-1} of σ-algebras on $[0,1)$; however it *is* the case that $\tilde{\mathcal{C}}_{-\infty} = \xi^{-1}(\mathcal{B}_\infty)$: "the remote past equals the remote future". Let $\tilde{Q}_n$ be the conditional expection on $\tilde{\mathcal{C}}_n$, and $\tilde{Q}_{-\infty}$ that on $\tilde{\mathcal{C}}_{-\infty}$. Then for all f in $\mathcal{L}^2(\mu)$, $\tilde{Q}_{-\infty}(f \circ \xi) = (P_\infty f) \circ \xi$.

The next lemma plays the role of lines 3 to 7 in the proof of Theorem 4 in [L].

2.4. LEMMA: *Given $\epsilon > 0$ and $f_0, ..., f_K$ in $\mathcal{L}^\infty(\tilde{\mu})$, then for all sufficiently large even integers J, $\tilde{Q}_{-(K+1/2)J}((f_0\tilde{S}^{-KJ})(f_1\tilde{S}^{-(K-1)J})\cdots(f_K))$ lies within ϵ of $(\tilde{Q}_{-\infty}(f_0)\tilde{S}^{-KJ})(\tilde{Q}_{-\infty}(f_1)\tilde{S}^{-(K-1)J})\cdots(\tilde{Q}_{-\infty}(f_K))$ in $\mathcal{L}^2(\tilde{\mu})$.*

Proof: Clearly it suffices to do this for $f_0, ..., f_K$ ranging over an $\mathcal{L}^2(\tilde{\mu})$-dense subset of the unit ball of $\mathcal{L}^\infty(\tilde{\mu})$; noting that $\bigvee_{N=0}^{\infty} \tilde{S}^{-N}\tilde{\mathcal{C}} = \tilde{\mathcal{A}}$, assume that each f_k is $\tilde{S}^{-N}\tilde{\mathcal{C}}$-measurable for a fixed positive integer N, and is in the aforementioned unit ball. Let $\delta = \epsilon/K$. Choose $J > 2N$, and so large that each $\tilde{Q}_{-J/2}f_k$ is within δ of $\tilde{Q}_{-\infty}f_k$ in $\mathcal{L}^2(\tilde{\mu})$. Write g_k for $(f_0\tilde{S}^{-(k-1)J})\cdots(f_{k-1})$, and B_k for $\tilde{Q}_{-(k+1/2)J}((f_0\tilde{S}^{-kJ})\cdots(f_k)) = \tilde{Q}_{-(k+1/2)J}((g_k\tilde{S}^{-J})(f_k))$. Each factor of g_k is $\tilde{S}^{-N}\tilde{\mathcal{C}}$-measurable, so $g_k\tilde{S}^{-J}$ is measurable with respect to $\tilde{S}^{-(N-J)}\tilde{\mathcal{C}} \subset \tilde{\mathcal{C}}_{-J/2}$, and by the basic properties of conditional expectation,

$$\tilde{Q}_{-J/2}((g_k\tilde{S}^{-J})(f_k)) = (g_k\tilde{S}^{-J})(\tilde{Q}_{-J/2}f_k).$$

But $\tilde{Q}_{-J/2}f_k$ is within δ of $\tilde{Q}_{-\infty}f_k$. So B_k is within δ of

$$\tilde{Q}_{-(k+1/2)J}((g_k\tilde{S}^{-J})(\tilde{Q}_{-\infty}f_k)) = \tilde{Q}_{(k+1/2)J}(g\tilde{S}^{-J})(\tilde{Q}_{-\infty}f_k).$$

One easily verifies that $\tilde{Q}_{-(k-1/2)J}(g_k)\tilde{S}^{-J}$ has the measurability and integration properties specifying the conditional expectation of $g_k\tilde{S}^{-J}$ on $\tilde{\mathcal{C}}_{-(k+1/2)J}$; thus this may be substituted in the previous equality, which therefore becomes

$$(\tilde{Q}_{-(k-1/2)J}(g_k)\tilde{S}^{-J})(\tilde{Q}_{-\infty}f_k) = (B_{k-1}\tilde{S}^{-J})(\tilde{Q}_{-\infty}f_k).$$

By induction, B_K is within $K\delta = \epsilon$ of $((\tilde{Q}_{-\infty}f_0)\tilde{S}^{-KJ})\cdots(\tilde{Q}_{-\infty}f_k)$. ∎

Note that the preceding lemma is actually a general fact: any ergodic probability- preserving endomorphism of positive entropy could have been used in the role of $\tilde{S}^{-1}$.

3. Proof of Proposition 1.1

Choose $\epsilon > 0$ and a positive integer R. Since finite linear combinations of exponentials are uniformly dense in $\mathcal{C}[0,1]$, it will suffice to find a positive integer b so that if for some positive integer m the number v/p^m is closer than 1 to b then $|\int \phi_r V d\mu| < 5\epsilon$ whenever $0 < |r| < R$.

Choose $\delta > 0$ so that if $0 < |r| < R$ then $|P_\infty \phi_r| < 1 - \delta$ on a set of measure $1-\epsilon$; this can be done by Lemma 2.3. Choose K so $(1-\delta)^{K/2} < \epsilon$. Choose an even integer J large enough for Lemma 2.4 to hold with $f_0 = \cdots f_K = \phi_r \circ \xi$ whenever $0 < |r| < R$; and also large enough that $2\pi r/p^{J/2} < \epsilon$. Let $b = p^J + \cdots + p^{(K+1)J}$, and let v and m be as in the statement of Proposition 1.1. Then $v = c + p^m b$ with $0 \le c < p^m$, so

$$\begin{aligned}\phi_r V &= \phi_{rv} \\ &= (\phi_{rc})(\phi_{rp^{m+J}})\cdots(\phi_{rp^{m+(K+1)J}}) \\ &= (\phi_{rc})(\phi_r S^{m+J})\cdots(\phi_r S^{m+(K+1)J}),\end{aligned}$$

and $\int \phi_r V d\mu = \int (\phi_{rc})(\phi_r S^{m+J})\cdots(\phi_r S^{m+(K+1)J}) d\mu$.

For $0 \le x < 1$ let τx be x truncated at the $(m + J/2)$th place in its expansion in powers of $1/p$; that is, $\tau x = x_1/p + \cdots + x_{m+J/2}/p^{m+J/2}$. Then $|x - \tau x| < p^{-(m+J/2)}$, so $|\phi_{cr}(x) - \phi_{cr}(\tau x)| \le 2\pi c|r||x-\tau x| < \epsilon$.Thus ϕ_{rc} is uniformly within ϵ of $\psi = \phi_{rc} \circ \tau$, for all such v and $0 < |r| < R$. So $|\int \phi_r V d\mu|$ is within ϵ of

$$|\int (\psi)(\phi_r S^{m+J})\cdots(\phi_r S^{m+(K+1)J}) d\mu|.$$

Lifting by ξ^{-1} to integrate over X rather than $[0,1)$ gives

$$|\int_X (\psi \circ \xi)(\phi_r S^{m+J} \circ \xi)\cdots(\phi_r S^{m+(K+1)J} \circ \xi) d\tilde{\mu}|.$$

Writing f for $\phi_r \circ \xi$, this may be rewritten as

$$|\int_X (\psi \circ \xi)(f\tilde{S}^{m+J}) \cdots (f\tilde{S}^{m+(K+1)J})d\tilde{\mu}|$$

$$= |\int_X (\psi \circ \xi\tilde{S}^{-(m+(K+1)J)})(f\tilde{S}^{-KJ}) \cdots (f)d\tilde{\mu}|.$$

Inserting a conditional expection inside the integral, this becomes

$$|\int_X \tilde{Q}_{-(K+1/2)J}((\psi \circ \xi\tilde{S}^{-(m+(K+1)J)})(f\tilde{S}^{-KJ}) \cdots (f))d\tilde{\mu}|.$$

Because of measurability of ψ in $\mathcal{P}_0^{m+J/2}$, $\psi \circ \xi$ is measurable in $\tilde{S}^{-(m+J/2)}\tilde{\mathcal{C}}$, and $\psi \circ \xi\tilde{S}^{-(m+(K+1)J)}$ is measurable in $\tilde{\mathcal{C}}_{-(K+1/2)J}$. Thus the latter function can be pulled past $\tilde{Q}_{-(K+1/2)J}$ in the previous expression, which then equals

$$|\int_X \psi \circ \xi\tilde{S}^{-(m+(K+1)J)}\tilde{Q}_{-(K+1/2)}((f\tilde{S}^{-KJ}) \cdots (f))d\tilde{\mu}|$$
$$\leq \int_X |\psi \circ \xi\tilde{S}^{-(m+(K+1)J)}\tilde{Q}_{-(K+1/2)}((f\tilde{S}^{-KJ}) \cdots (f))|d\tilde{\mu}$$
$$\leq \int_X |\tilde{Q}_{-(K+1/2)J}((f\tilde{S}^{-KJ}) \cdots (f))|d\tilde{\mu}.$$

By choice of J, Lemma 2.4 implies that this differs by less than ϵ from

$$\begin{aligned}\int_X |((\tilde{Q}_{-\infty}f)\tilde{S}^{-KJ}) \cdots (\tilde{Q}_{\infty}f)|d\tilde{\mu} &= \int_X |(P_\infty(\phi_r) \circ \xi\tilde{S}^{-KJ}) \cdots ((P_\infty\phi_r) \circ \xi)|d\tilde{\mu} \\ &= \int_X |((P_\infty\phi_r) \circ \xi) \cdots ((P_\infty\phi_r) \circ \xi\tilde{S}^{KJ})|d\tilde{\mu} \\ &= \int_X |((P_\infty\phi_r) \circ \xi) \cdots (((P_\infty\phi_r)S^{KJ}) \circ \xi)|d\tilde{\mu} \\ &= \int |(P_\infty\phi_r) \cdots ((P_\infty\phi_r)S^{KJ})|d\mu.\end{aligned}$$

Since each $|(P_\infty\phi_r)S^{KJ}| \leq 1 - \delta$ on a set of measure at least $1 - \epsilon$, Lemma 2.2 tells us that this is $< 2\epsilon + (1-\delta)^{K/2} < 3\epsilon$, so $|\int \phi_r V d\mu| < 5\epsilon$. ∎

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